

On the allocation of multiple divisible and non-transferable commodities

Ephraim Zehavi¹, *Fellow, IEEE*, and Amir Leshem¹, *Senior Member, IEEE*

Abstract

When there is a dispute between players how to divide multiple divisible commodities, how should it be resolved? In this paper we introduce a multi-commodity game model. This model enables cooperation between multiple players to bargain on sharing K commodities, when each player has a different value for each commodity. It thus extends the sequential discrete Raiffa solution and the Aumann bankruptcy solution to multi-commodity cases.

Keywords: Spectrum optimization , distributed coordination, game theory, Raiffa bargaining solution, interference channel, multiple access channel.

I. INTRODUCTION

In the last thirty years there has been extensive exploration of the axiomatic bases of bargaining solutions and ways to resolving conflict claims in bankruptcy cases. In this paper we extend two cooperative bargaining solutions, the discrete Raiffa bargaining solution [1] and the Aumann Bankruptcy (AB) [2] for resolving the allocation of K commodities to N players. In the discrete Raiffa solution the players reach an agreement step by step on an intermediate partition of the utility. However, if some utility is left over, all the players continue to solve the problem until Pareto optimality is achieved. The Aumann Bankruptcy solution is based on an extension of a Talmudic approach involving two individuals claiming a single garment, to resolve a dispute between heirs.

In the literature there are several alternative approaches to analyzing collaborative solutions. One approach is based on building an axiomatic structure that leads to a single solution. Other approaches emphasize the negotiation process to reach a final agreement. Salonen [3] was the first to establish a step-by-step axiomatic definition to the discrete Raiffa solution for the N -player bargaining problem, based on four axioms. Livne [4], as well as Peters and van Damme [5] presented characterizations of the continuous Raiffa solution. Recently, Trocket [6] suggested viewing the discrete Raiffa solution as a repetition of a process based on three standard axioms; namely (a) Pareto optimality (b) invariance to

¹Faculty of Engineering, Bar-Ilan University, Ramat-Gan, 52900, Israel. e-mail: ephraim.zehavi@eng.biu.ac.il .

affine transformation, and (c) symmetry. Diskin and et al. [7] generalized the Raiffa solution to the case of multi players achieving interim settlements step-by-step. They defined a family of discrete solutions for N -person bargaining problems which approaches the continuous Raiffa solution as the step size gradually becomes smaller. Anbarci and Sun [8] proposed a unified framework for characterizations of different axioms that lead to different bargaining solutions.

Another approach is to define a bargaining process that leads to a specific bargaining solution. Myerson [9], Tanimura and Thoron [10] and Trockel [11] proposed a mechanism for reaching bargaining solutions. The mechanism allows two players to make a sequence of simultaneous propositions and to converge to the discrete Raiffa solution.

The Aumann and Maschler [2] bankruptcy solution is based on an interpretation of two bargaining scenarios discussed in the Talmud. The first case is the Contested Garment (CG) problem where two men disagree on the ownership of a garment¹. The second case² addresses the estate division problem between three women. Aumann and Maschler [2] constructed two rules that generalize the CG and can be applied to resolve the bankruptcy problem. Thomson [12], [13] provides a broad summary on progress in the last forty years on the rules and axioms for resolving bankruptcy.

In many problems of dividing multiple commodities one can sell the commodities and divide the money received among the participants. This is indeed the case in bankruptcy problems. Auctioning each asset results in the highest value for the debtors. By contrast, there are cases where resources are shared for which an auction is not an option and the utility is inherently non-transferable, and each player has a different utility for each resource. For example, when an operator is allocating wireless communication frequencies to customers under a fixed price best effort contract, the utility for the customers is non-transferable, since each frequency channel has different value to each customer, but there is no exchange of utility possible between the customers. For these cases of non-transferable utility, the fair allocation problem becomes much more complicated. Previously, we have examined the computation of the Nash bargaining solution [14], and the Kalai Smorodinski solution [15] for this problem. We also proved [16] that for any Pareto optimal allocation of K resources to N players ($K > N$), there is a need to share at most $N - 1$ resources among the players, and the other resources are allocated to a single player. In this paper we focus on the Raiffa and Auman solutions and propose an efficient resource allocation algorithm

¹ Mishna Baba Metzia 2a: The first man claimed half of it belongs to him and the other claimed it all; the decision was that the one who claimed half is awarded $1/4$ and the other is awarded $3/4$. The principle is clear: the first man agrees that half of the garment does not belong to him. Therefore, the bargaining is only on half of the garment.

² Kethubot 93a: a man married three women. The first woman had a marriage contract of 100, the second of 200, and the third of 300. The man dies and his estate is worth E . The ruling of Rabbi Nathan was as follows: If the estate is worth $E = 100$, then the estate will be divided equally, namely $33\frac{1}{3}$ for each. If the estate is worth 200 the division will be (50, 75, 75) and if it is worth $E = 300$ the division is (50, 100, 150), respectively.

for the case of non-transferable utility.

Marmol and Ponsati [17] addressed a similar problem of resolving global bargaining problems over a finite number of different issues. They defined max-min and leximin global bargaining solutions. Zehavi and Leshem [16] proved that the number of commodities that have to be shared by more than one players is always less than the number of players whenever the utility functions are additively separable across commodities. The case where $N = 2$ and K was solved using the Nash Bargaining Solution (NBS) [18], [19], [20], and [21], the Perles-Maschler solution and the discrete Raiffa solution in [21]. Leshem and Zehavi introduced NBS for the general case of N players and K commodities in [14], and for the Kalai-Smorodinsky solutions (KSS) in [15].

Extending the Raiffa solution and Aumann bankruptcy solution to the case of N players that have to share K commodities can also be viewed as a bankruptcy problem where the number of commodities are K and the commodities have a different value for each player. In Section II we define the model with N players for the discrete Raiffa bargaining solution and the Aumann bankruptcy solution. Section III extends the solutions to the case of N players and K commodities and proves that the solutions are ϵ -PO. In section IV we apply these solutions to two examples with three players and seven commodities.

II. SINGLE-COMMODITY BARGAINING GAME

An N -player single-commodity bargaining game is described as the set of players, $\mathcal{N} = \{1, \dots, N\}$, where each player has access to a single commodity. The utility of the commodity to the n 'th player is u_n . We define a pair (\mathbf{S}, \mathbf{d}) , where $\mathbf{S} \in \mathcal{R}^{\mathcal{N}}$ is a compact, convex, and comprehensive set of all possible outcomes of the bargaining, and \mathbf{d} is the players' possible disagreement point. If players do not reach an agreement, the utility of player n will be d_n . Thus the point of disagreement is $\mathbf{d} = (d_1, d_2, \dots, d_N)$. The following definitions will be used in what follows.

Definition 2.1: The individually rational part of \mathbf{S} is all the points $s \in \mathbf{S}$ that provide a higher utility than the disagreement utility to all players, i.e $\mathbf{S}_d = \{\mathbf{x} = (s_1, \dots, s_N) | \mathbf{x} \geq \mathbf{d}, \mathbf{x} \in \mathbf{S}\}$.

Players will agree to negotiate only if they can get more than their disagreement point.

Definition 2.2: The ideal point of player n is $I_n(\mathbf{S}_d, \mathbf{d}) = \max \{\mathbf{x}_n \in \mathbf{R} | \mathbf{x} \in \mathbf{S}_d\}$, and the ideal point vector for the n players is the vector $\mathbf{I}(\mathbf{S}_d, \mathbf{d}) = \{I_n(\mathbf{S}_d, \mathbf{d})\}_{n=1}^N$.

In other words, the ideal point for player n is the maximal utility that a player n can get, when any other player n' only gets a utility $d_{n'}$.

Definition 2.3: The Midpoint (MP) Rule:

The midpoint is the mapping $\mu : \mathbf{S}_d \rightarrow \mathcal{R}^{\mathcal{N}}$, and

$$\begin{aligned} \mu(\mathbf{S}_d, \mathbf{d}) &\stackrel{def}{=} \mathbf{m} = \frac{1}{N} \mathbf{I}(\mathbf{S}_d, \mathbf{d}) + (1 - \frac{1}{N}) \mathbf{d}, \\ \mathbf{m} &= \{m_n\}_{n=1}^N, \end{aligned} \tag{1}$$

where $m_n = \frac{1}{N}I_n(\mathbf{S}_d, \mathbf{d}) + (1 - \frac{1}{N})d_n$ is the midpoint for player n .

Note that the convexity of \mathbf{S}_d implies that the midpoint \mathbf{m} is always in \mathbf{S}_d .

Definition 2.4: A bargaining solution is a mapping: $L : \mathbf{S}_d \rightarrow \mathcal{R}^N$ and $L(\mathbf{S}_d, \mathbf{d}) \in \mathbf{S}$.

Definition 2.5: Let $\mathbf{S} \subset \mathcal{R}^N$ be a set. Then $\mathbf{s} \in \mathbf{S}$ is Pareto efficient if there is no $\mathbf{x} \in \mathbf{S}$ for which $x_i > s_i$ for all $i \in N$; $\mathbf{s} \in \mathbf{S}$ is strongly Pareto efficient if there is no $\mathbf{x} \in \mathbf{S}$ for which $x_i \geq s_i$ for some $i \in N$. The Pareto frontier is defined as the set of all $\mathbf{s} \in \mathbf{S}$ that are Pareto efficient, and is denoted by $\partial\mathbf{S}$.

Definition 2.6: Let $\mathbf{S} \subset \mathcal{R}^N$ be a set, and $\partial\mathbf{S}$ is the Pareto frontier of the set. Then, $\mathbf{x} \in \mathbf{S}$ is ϵ -Pareto efficient if there is $\mathbf{s} \in \partial\mathbf{S}$ for which $|x_i - s_i| < \epsilon$ for all $i \in N$.

We now define the Discrete Raiffa Bargaining Solution (DRB) and the Aumann Bankruptcy Solution (ABS) for a single commodity.

A. Discrete Raiffa bargaining solution

The Raiffa procedure is a step-by-step process where each step increases the utility of all players. In the first step, the players agree on the first partial division of the commodity, where each player gets his midpoint of the set \mathbf{S} , i.e. the n 'th player gets $m_n^1 = \frac{1}{n}I_n(\mathbf{S}, \mathbf{d}) + (1 - \frac{1}{n})d_n$. In the next steps players will bargain on the remainder of the commodity. If the players fail to come to an agreement in step $j + 1$, then the final outcome of step j will be the final agreement of the whole game. Thus, the agreement of the j -th step is the disagreement point (or the threat point) for the $j + 1$ step.

The discrete Raiffa solution procedure is shown in Table I.

Example 1 : Consider the convex set \mathbf{S} as depicted in Figure 1,

$$\mathbf{S} = \left\{ (u_1, u_2) : \begin{array}{l} u_1 \geq 0, u_2 \geq 0, \\ u_2 \geq 160 - \frac{u_1}{3}, \\ u_2 \geq 150 - \frac{5}{2}(u_1 - 30), \\ u_2 \geq 100 - 5(u_1 - 50) \end{array} \right\}.$$

Assume that the point of disagreement is $\mathbf{d} = (0, 0)$. Thus, the ideal point for player 1 is 70 and for player 2 is 160. The midpoint of the the first step is $\mathbf{m}_1 = (35, 80)$. The disagreement point of the second step is $\mathbf{d} = \mathbf{m}_1$. Therefore, the individual rational set of \mathbf{S} induced by $\mathbf{d} = \mathbf{m}_1$ is the set

$$\mathbf{S}_{\mathbf{m}_1} = \left\{ (u_1, u_2) : \begin{array}{l} u_1 \geq 35, u_2 \geq 80, \\ u_2 \geq 150 - \frac{5}{2}(u_1 - 30) \\ u_2 \geq 100 - 5(u_1 - 50) \end{array} \right\}.$$

In the second step we get,

$$\mathbf{m}_2 = \mu(\mathbf{S}_{\mathbf{m}_1}, \mathbf{m}_1) = (44.5, 108, 75),$$

and

$$\mathbf{S}m_2 = \left\{ (u_1, u_2) : \begin{array}{l} u_1 \geq 44.5, u_2 \geq 108.75 \\ u_2 \geq 150 - \frac{5}{2}(u_1 - 30) \end{array} \right\}.$$

Now in the third step we get

$$\mathbf{m}_3 = \mu(\mathbf{S}\mathbf{m}_2, \mathbf{m}_2) = \mathbf{S}\mathbf{m}_3 = (45.5, 111.25).$$

The set S and the steps of the bargaining process are depicted in Table II and in Figure 1.

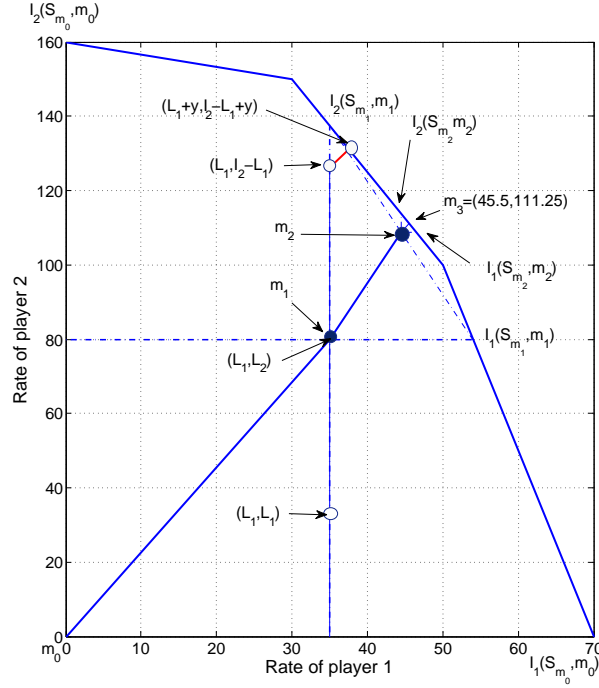


Fig. 1. The Raiffa and Aumann bargaining solutions for the 2-players game.

B. The Aumann bankruptcy solution

Aumann and Maschler [2] considered the problem of the division of a property E , when the creditors have debts c_1, \dots, c_n , that are worth more than E . They proposed allocating the property according to an extension of the Talmud rule known as the Contested Garment.

Definition 2.7: Contested Garment, (CG) Rule:

Two creditors have claims c_1 and c_2 on a property E . Then, the amount that creditor i will be awarded is

$$x_i = \frac{E - (E - c_1)_+ - (E - c_2)_+}{2} + (E - c_{3-i})_+, i = 1, 2,$$

TABLE I
DISCRETE RAIFFA BARGAINING SOLUTION FOR N PLAYERS

Initialization: $\mathbf{m}_0 = \mathbf{d}, \mathbf{S}_{\mathbf{m}_0} = \mathbf{S}$. Compute for each player n , the ideal point $I_n(\mathbf{S}_{\mathbf{m}_0}, \mathbf{m}_0)$, Set: $j = 0, Error = N$.
While $Error \geq \epsilon$ $\mathbf{m}_{j+1} = \mu(\mathbf{S}_{\mathbf{m}_j}, \mathbf{m}_j)$, see equation (1), $Error = \min_{\mathbf{s} \in \partial \mathbf{S}} \ \mathbf{s} - \mathbf{m}_{j+1}\ $, Set $j = j + 1$. End
Allocate the utility according to \mathbf{m}_j to the players. End

TABLE II
RAIFFA SOLUTION FOR 2 PLAYERS

Initial Status	\mathbf{m}_0	Ideal point
Player 1	0	70
Player 2	0	160
First step	\mathbf{m}_1	Ideal point
Player 1	35	54
Player 2	80	137.5
Second step	\mathbf{m}_2	Ideal point
Player 1	44.5	46.5
Player 2	108.75	113.75
Third step	\mathbf{m}_3	Final point
Player 1	45.5	45.5
Player 2	111.25	111.25
Aumann		Final point
Player 1	$38\frac{4}{7}$	$38\frac{4}{7}$
Player 2	$128\frac{4}{7}$	$128\frac{4}{7}$
NBS		Final point
Player 1	45	45
Player 2	112.5	112.5
KS		Final point
Player 1	47	47
Player 2	107.5	107.5

where, $(x)_+ = \max(x, 0)$. We denote this division as $CG(c_1, c_2, E) = (x_1, x_2)$.

Assume that the debts are ranked in increasing order: $c_1 \leq \dots \leq c_n$, and $C = \sum_{n=1}^N c_n$. Aumann proposed the following allocation: each creditor will get x_n , where

$$x_n = \begin{cases} \min \left\{ \frac{c_n}{2}, \lambda \right\} & E \leq \frac{1}{2}C, \\ \max \left\{ \frac{c_n}{2}, c_n - \mu \right\} & E > \frac{1}{2}C, \end{cases} \quad (2)$$

and λ and μ are chosen to satisfy the constraint $\sum_{n \in \mathcal{N}} c_n = E$.

It is easy to verify that the allocation of x_i and x_j to players i and j , respectively, satisfies the CG rule. Aumann suggested interpreting the Talmud rule as a composition of the following two rules:

Definition 2.8: Constrained equal-awards rule- $E \leq \frac{1}{2}C$ When the value of the property is less than half of the sum of debts, no creditor will be awarded more than half of his debt. Any two creditors i and j will be awarded x_i and x_j , such that $CG(c_i, c_j, E_{i,j}) = (x_i, x_j)$, $E_{i,j} = x_i + x_j$, $\forall i, j$ and $\sum_{n=1}^N x_n = E$.

Definition 2.9: Constrained equal-losses rule- $E > \frac{1}{2}C$ When the value of the property is larger than half of the sum of debts, no creditor will lose more than half of his debt. Any two creditors i and j will lose x_i and x_j , such that $CG(c_i, c_j, E_{i,j}) = (x_i, x_j)$, $E_{i,j} = x_i + x_j$, $\forall i, j$ and $\sum_{n=1}^N x_n = C - E$.

The allocation can be resolved using an algorithm with a complexity of $O(\log n)$.

We now modify the bankruptcy solution and apply it to the bargaining problem. Let us adopt the following modification:

- The maximum utility that a player n claims is $I_n(\mathbf{S}_d, \mathbf{d})$, and without an agreement he gets d_n . Therefore, the negotiation is only on the surplus $c_n = I_n(\mathbf{S}_d, \mathbf{d}) - d_n$. For simplicity of notation, we will use I_n to denote $I_n(\mathbf{S}_d, \mathbf{d})$, and D for $\sum_{n=1}^N d_n$.
- In contrast to the bankruptcy problem, there is no single property with a value of E that has to be divided between the players. Here, the solution has to be on the Pareto frontier of set \mathbf{S} . Therefore, the CG rule for the bargaining problem will be $CG(c_i, c_j, E_{i,j}) = (x_i, x_j)$, $E_{i,j} = x_i + x_j$, and either $(x_1 + d_1, \dots, x_N + d_N) \in \partial \mathbf{S}$ or $(I_1 - x_1, \dots, I_N - x_N) \in \partial \mathbf{S}$.

The modified algorithm is given in Table III. We will use the Kaminsky [22] water filling interpretation to describe the algorithm. Figure 2 depicts N containers of differing sizes, representing the claims of the players, into every one of which we pour water representing the utility. A container representing the claim of a player n is divided into two halves connected by a narrow tube that allows the water to run through it, but with almost zero capacity. All containers are connected at level $L_0 = 0$ by a tube that likewise is very narrow but allows the fluid to pass between containers according to the law of communicating containers. All containers are at the same height above the ground, L_0 , and width and have a different tube. The container with the smallest capacity has the longest tube. The lower part of the container has a capacity that is equal to half of the claim of player- $L_n = (I_n - d_n)/2, n \in \mathcal{N}$ above level L_0 , plus what

TABLE III
AUMANN BARGAINING SOLUTION FOR N PLAYERS

<p>Initialization: If $\left\{ \frac{I_1+d_1}{2}, \dots, \frac{I_N+d_N}{2} \right\} \notin \mathbf{S}_d$ then go to A, otherwise go to B.</p>
<p>A. All players lose at least half of the debt.</p> <p>1. Do a binary search to find the smallest p such that $\{a_1, \dots, a_N\} \notin \mathbf{S}_d$, where</p> $a_n = \begin{cases} \frac{I_n+d_n}{2} & n \leq p \\ \frac{I_p-d_p}{2} + d_n & p < n \leq N \end{cases} .$ <p>2. Do a binary search to find y s.t.</p> $\{b_1, \dots, b_N\} \in \partial \mathbf{S},$ $b_n = \begin{cases} d_n + y & p = 1 \\ \frac{I_n+d_n}{2} & n < p < N \\ \frac{I_p-d_p}{2} + d_n + y & p \leq n \leq N \end{cases} ,$ <p>Allocate the utility b_n to player n.</p> <p>Exit</p>
<p>B. All players lose at most half of the debt.</p> <p>1. Do a binary search to find the smallest p such that $\{a_1, \dots, a_N\} \in \mathbf{S}_d$,</p> $a_n = \begin{cases} \frac{I_n+d_n}{2} & n \leq p \\ I_n - \frac{I_p-d_p}{2} & p < n \leq N \end{cases} .$ <p>2. Do a binary search to find y such that</p> $\{b_1, \dots, b_N\} \in \partial \mathbf{S},$ $b_n = \begin{cases} \frac{I_n+d_n}{2} & n < p \\ I_n - \frac{I_p+d_p}{2} + y & p \leq n \leq N \end{cases} .$ <p>Allocate the utility b_n to player n.</p> <p>Exit</p>

he can get by competition- d_n . The upper part has a capacity equal to half of the claim of the player. Thus, the capacity of a container represents the player's claim plus d_n . The containers are ranked according to their claims.

We now pour water into all the containers. If the extra utility to be shared is between D and $D + N \cdot L_1$ then all the containers (players) share the water (extra utility) equally, and the water level in all the containers is the same according to the law of communicating containers. If the extra utility is greater than $D + N \cdot L_1$, then the container (player) with the smallest (volume) claim stops receiving anything for a while, and the water is divided equally among all the other containers until each container has an amount equal to the second smallest half-claim L_2 plus d_n . This process continues as follows: whenever the water level is above L_p player p stops receiving anything, while the rest of the players share the water (extra utility) equally. Therefore, whenever the extra utility to be shared by the players is smaller than the

half-sum of the claims plus D , i.e. $\sum_{n=1}^N L_n + d_n$, each player receives at most his half claim according to the constrained equal awards rule.

When the extra utility exceeds half the sum of the claims, the calculation is made in accordance with each player's losses: the difference between the player's claim $\frac{1}{2}(I_n - d_n)$ and what he actually gets is b_n . Now, if the water level is between L_{2N-p+1} , and L_{2N-p} , $p \in \mathcal{N}$, the water is shared equally between the p 'th container and N 'th container according to constrained equal losses rule.

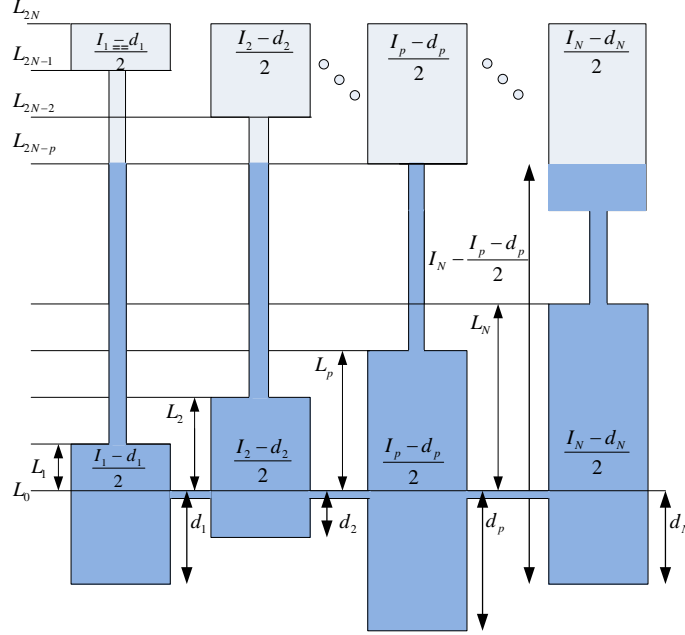


Fig. 2. Water filling interpretation of the Aumann bargaining solution.

Example II: Assume the same conditions as in Example I. The ideal point for player 1 and player 2 is 70 and 160, respectively, and both claim to get the utility of the ideal point. An allocation of half of the claim for each player is also inside the set, $(L_1, L_2) = (35, 80) \in \mathcal{S}$. The point $(L_1, L_2 - L_1) = (35, 125)$ is also inside \mathcal{S} , so the solution is to share the remaining utility equally between the players. Hence, we have to search for a point $(u_1, u_2) = (35 + y, 125 + y)$ that is on the Pareto frontier of set \mathcal{S} , namely $u_2 = 150 - \frac{5}{2}(u_1 - 30)$. Thus, point $(L_1 + y, L_2 - L_1 + y) = (38\frac{4}{7}, 128\frac{4}{7})$ provides equal losses to both players. These points are marked in Figure 1.

Note that in the case of 2 players the Aumann bargaining solution always operates according to the constrained equal losses rule (due to the convexity of set \mathcal{S}). It is easy to show that the player with the larger claim gets more than in the Raiffa bargaining solution.

III. EXTENSION TO K COMMODITIES

In this section we extend the Raiffa and Aumann bargaining solutions to a multi-commodity scenario. We first define the game and two rules. The rules define functions that assign a unique solution or a set

of equivalent solutions to each bargaining problem (in a way that will be defined below).

Definition 3.1: A multi-commodity bargaining problem: Assume that there is a set of players $\mathcal{N} = \{1, \dots, N\}$, where each player has access to $\mathcal{K} = \{1, \dots, K\}$ commodities. The utility of the k 'th commodity to the n 'th player is u_{nk} , and the utility functions are additively separable across commodities. A player's n 'th action is a vector $\alpha_n = (\alpha_{n1}, \dots, \alpha_{nK})^T$. We require that $0 \leq \alpha_{nk} \leq 1$ and $\sum_{k=1}^K \alpha_{nk} = 1$. The utility vector of player n is $\mathbf{u}_n(\alpha_n) = (\alpha_{n1}u_{n1}, \dots, \alpha_{nK}u_{nK})^T$ and his total utility is $u_n = \sum_{k=1}^K \alpha_{nk}u_{nk}$. Let $\mathbf{d} = (d_1, d_2, \dots, d_N)$, where $d_n = \sum_{k=1}^K d_{nk}$ is the disagreement point of the n 'th player. Assuming a pair (\mathbf{S}, \mathbf{d}) is defined as follows:

$$(\mathbf{S}, \mathbf{d}) = \left\{ (\mathbf{u}_1, \dots, \mathbf{u}_N) : \begin{array}{l} u_n \leq d_n, \forall 1 \leq n \leq N, \\ \text{there is a Matrix } \mathbf{A} \\ \text{such that:} \\ \mathbf{A} = \{\alpha_n\}_{n=1}^N, \\ \sum_{n=1}^N \alpha_{nk} = 1, \forall k, \\ \alpha_{nk} \geq 0, \forall n, k, \\ \sum_{k=1}^K \alpha_{nk}u_{nk} = u_n, \forall n. \end{array} \right\}. \quad (3)$$

The pair (\mathbf{S}, \mathbf{d}) , $\mathbf{S} \in \mathcal{R}^{\mathcal{N}\mathcal{K}}$, is a compact, convex and comprehensive set of all possible outcomes of the bargaining. If the players do not reach an agreement, the utility of the k 's commodity to player n will be d_{kn} .

Note that in a multi-commodity game the interest of the players is to maximize the sum of the utilities, u_n . The way in which the commodities are combined in the allocation makes no difference to the players as long as the selection results in maximum total utility. Solutions that allocate the same utility to each player are equivalent solutions. The objective of the game is to find the point on the Pareto frontier of the set \mathbf{S}_d . More formally we use the following definitions:

Definition 3.2: Additive utility space: The additive utility space $\mathbf{S}^N \in \mathcal{R}^N$ is defined by the many-to-one mapping $\mathbf{U} : \mathbf{S} \rightarrow \mathcal{R}^N$, and $\mathbf{U}(\{\alpha_n\}_{n=1}^N) = \{u_1, \dots, u_N\}$.

Definition 3.3: Restricted Ideal Point (RIP) rule: For every bargaining problem (\mathbf{S}, \mathbf{d}) , $\mathbf{d} = \{d_1, \dots, d_N\}$, the ideal point for player n is the allocation that maximizes his own utility while maintaining the allocation $d_{n'}$ for every player $n' \neq n$. i.e., I_n . i.e.,

$$I_n(\mathbf{S}, \mathbf{d}) = \max \left\{ u_n : \begin{array}{l} \{u_1, \dots, u_N\} \in (\mathbf{S}, \mathbf{d}) \\ \text{and} \\ u_p = d_p, \forall p \neq n, \\ u_n(\mathbf{S}, \mathbf{d}) = \sum_{k=1}^K \alpha_{nk}u_{nk} \end{array} \right\}. \quad (4)$$

Each player has a different ideal point and a different allocation. The ideal point for each player can be posed as a linear programming problem, where player n selects the best assignment matrix $\mathbf{A}^{(n)} = (\alpha_1^{(n)}, \dots, \alpha_N^{(n)})$ that maximizes the assigned commodities to him.

$$\begin{aligned}
& \mathbf{max} \quad \sum_{k=1}^K \alpha_{nk}^{(n)} u_{nk} \\
& \mathbf{subject\ to:} \quad \sum_{n=1}^N \alpha_{nk}^{(n)} = 1, \forall k, \\
& \quad \alpha_{nk}^{(n)} \geq 0, \forall n, k, \\
& \quad \sum_{k=1}^K \alpha_{pk}^{(n)} u_{pk} = u_p, \forall p \neq n. \\
& \quad \sum_{k=1}^K \alpha_{pk}^{(n)} u_{pk} \geq u_n.
\end{aligned} \tag{5}$$

Lemma 3.1: Let's assume a bargaining problem (\mathbf{S}, \mathbf{d}) , with the ideal points $\{I_n(\mathbf{S}, \mathbf{d})\}_{n=1}^N$ and the set of allocations, $\{\mathbf{A}_n\}_{n=1}^N$. Then, the midpoint for player n is uniquely defined by the allocation matrix $\mathbf{A} = \frac{1}{N} \sum_{p=1}^N \mathbf{A}^{(p)}$ and is given by $m_n = \sum_{k=1}^K A_{nk} u_{nk}$.

Proof: The mid point for player n is equal to $m_n = \frac{1}{N} I_n(\mathbf{S}_d, \mathbf{d}) + (1 - \frac{1}{N}) d_n$,

$$\begin{aligned}
m_n &= \sum_{k=1}^K A_{nk} u_{nk} = \frac{1}{N} \sum_{p=1}^N \sum_{k=1}^K A_{nk}^{(p)} u_{nk} \\
&= \frac{1}{N} \sum_{k=1}^K A_{nk}^{(n)} u_{nk} + \frac{1}{N} \sum_{p=1, p \neq n}^N \sum_{k=1}^K A_{nk}^{(p)} u_{nk} \\
&= \frac{1}{N} I_n + \frac{N-1}{N} d_n
\end{aligned} \tag{6}$$

The set \mathbf{S} is convex and all ideal points are on the Pareto frontier of the set. Therefore, the mid point vector $\mathbf{m} = \{m_n\}_{n=1}^N$, and the allocation matrix \mathbf{A} defines uniquely a point in the set (\mathbf{S}, \mathbf{d}) .

A. The Raiffa bargaining solution for a multi-commodity Bargaining Game

The Raiffa bargaining solution for a K commodities bargaining game is based on iterations of the two rules MP and RIP. The bargaining is done by step by step, where agreement on the current step becomes the point of disagreement for the next step. In the initial step the ideal point $I_n(\mathbf{S}_d, \mathbf{d}) = u_n$, and the midpoints $m_n^{(1)} = \frac{1}{N} u_n + (1 - \frac{1}{N}) d_n$ are computed for each player. Now, at each step j we first apply the RIP rule N times to find the ideal point for all players and the assignment matrices that can guarantee a midpoint for each player. In our case this step requires solving N linear programming problems.

$$\begin{aligned}
& \mathbf{max} \quad \sum_{k=1}^K \alpha_{nk}^{(n)} u_{nk} \\
& \mathbf{subject\ to:} \quad \sum_{n=1}^N \alpha_{nk}^{(n)} = 1, \forall k, \\
& \quad \alpha_{nk}^{(n)} \geq 0, \forall n, k, \\
& \quad \sum_{k=1}^K \alpha_{pk}^{(n)} u_{pk} = m_p^j, \forall p \neq n. \\
& \quad \sum_{k=1}^K \alpha_{pk}^{(n)} u_{pk} \geq m_n^j.
\end{aligned} \tag{7}$$

TABLE IV

RAIFFA BARGAINING SOLUTION PROCEDURE FOR THE MULTI-COMMODITY RESOURCE ALLOCATION PROBLEM

Initialization: Set $\mathbf{m}_0 = \{d_1, \dots, d_n\}, j = 0$. $\Delta = K, \forall n \in \{0, \dots, N\}$. Set $\epsilon = 10^{-4}$.
Computation : While $\Delta \geq \epsilon$ Set $j = j + 1$. for $n=1:N$ Find the ideal point, I_n^j according to the LP in (7). Set the initial midpoint for player n : $m^{j+1} = I_n^j/N + (1 - \frac{1}{N})d_n$. $m_n = m_n^j$ end Set: $\Delta = N \max_n \ I_n^j - m_n\ _2$ End
Allocate to player n the commodities according to α 's, $\{\alpha_{nk}^j\}$, and the final utility of player n is $u_n = \sum_1^K \alpha_{nk}^j u_{nk} = m_n$. End

Then, we apply the MP rule to get the next midpoint vector, \mathbf{m}^{j+1} , for the next step. We repeat these steps until the distance from the Pareto frontier is arbitrarily small. The algorithm is shown in Table IV.

Lemma 3.2: The above procedure converges to a ϵ -Pareto optimal solution.

Proof: Assuming that the procedure stops when $\max_n |I_n^j - m_n| \leq \epsilon$, and the final allocation matrix is $\mathbf{A} = \frac{1}{N} \sum_{p=1}^N \mathbf{A}^{(p)}$. Then the final utility for player n is m_n , according to (7). Let $\mathbf{I}_s(\mathbf{S}, \mathbf{d}) = \{I_1^j, m_2, \dots, m_N\}$ be a point on $\partial \mathbf{S}$. We have to prove that the point \mathbf{m} is at a distance of less than ϵ from the Pareto frontier. The distance of \mathbf{m} from the Pareto frontier is bounded by

$$\|\mathbf{I}_s(\mathbf{S}, \mathbf{d}) - \mathbf{m}\|_2 \leq \max_n |I_n^j - m_n| \leq \epsilon.$$

Therefore, the procedure converges to a ϵ -Pareto optimal solution. ■

Example III: Assume that we have two players and three commodities as shown in Table V. The values of utilities A , B , and C for player 1 are 20, 20, and 30, and for player 2 are 100, 50, and 10, respectively. The achievable utility region between the players are inside set \mathbf{S} as is depicted in Figure 1. The final agreement is reached after three steps. There may be multiple options in the utility space that provide the same utility in the intermediate steps for sharing the multi-commodities. For example, in the first step of the discrete Raiffa bargaining solution the allocation to the first player is $\alpha_{11} = 0, \alpha_{12} = 0.725, \alpha_{13} = 1$ and to the second player $\alpha_{21} = 1, \alpha_{22} = 0.25, \alpha_{23} = 0$, provides the first player a utility of 44.5 and to the second

TABLE V
RAIFFA SOLUTION FOR 3 PLAYERS

Initial Status	A	B	C	Ideal Utility
Player 1	20	20	30	70
Player 2	100	50	10	160
First Run	k=1	k=2	k=3	Ideal Utility
Player 1	0	5	30	19
Player 2	80	0	0	57.5
Second Run	A	B	C	Ideal Utility
Player 1	0	14.5	30	2
Player 2	100	8.75	0	5
Third Run	A	B	C	Final I Utility
Player 1	0	15.5	30	45.5
Player 2	100	11.25	0	111.25
Aumann	A	B	C	Final Utility
Player 1	0	8.6	30	38.6
Player 2	100	28.6	0	128.6

player a utility of 108.75, respectively. The same utilities can also be obtained if the allocation for the first player is $\alpha_{11} = 0.1, \alpha_{12} = 0.625, \alpha_{13} = 1$ and to the second player $\alpha_{21} = 0.9, \alpha_{22} = 0.375, \alpha_{23} = 0$. However, the final allocation is unique.

Lemma 3.3: At every step of the DRB the unallocated utility is reduced by a factor of $(1 - \frac{1}{N})^N > e^{-1}$.

Proof: Any allocation of commodities by a matrix A can be mapped as a point in the additive utility space \mathcal{R}^N . A new disagreement point reduces the distance between the ideal point and the previous disagreement point by a factor of $(1 - \frac{1}{N})$, in every coordinate. Therefore, in every step of the DRB the unallocated utility is reduced by a factor of $(1 - \frac{1}{N})^N$. ■

The Raiffa solution is obtained by solving series of linear programming problems. The KKT conditions and the properties of the solution are derived in Appendix A in the general case. The solution for the case of two players is given in Appendix B. Several comments are in order:

- If $d_n < \frac{u_n}{N}, \forall N$, then there is a solution to the bargaining problem. This is because if $1/N$ of each commodity is allocated to each player, then the utility allocated to any player will be greater than what he can get by disagreement.
- Set S , is constructed by a finite number of intersections of hyper-planes. Every intermediate disagreement reduces the number of hyper-planes that define the Pareto frontiers of the set. Therefore, the final bargaining solution is achieved in finite steps, if the Raiffa solution is not on the intersection line of two hyper-planes.

B. Extension of the Aumann bargaining solution to the multi-commodity game

The extension of the Aumann Bargaining solution to the multi-commodity case is based on a binary search of a Pareto optimal allocation that satisfies the CG rule and the RIP rule. With no loss of generality, assume that the players are ranked such that $u_n \leq u_p$, implies that $n \leq p, \forall n, p$. The CG rule defines $2N + 1$ levels, L_n (see Figure 2), where each level corresponds to a point in the utility space, $\mathcal{R}^{\mathcal{N}\mathcal{K}}$ that can be either inside set $\mathcal{S}_d \in \mathcal{R}^{\mathcal{N}\mathcal{K}}$ or outside. The bargaining solution has to be on the Pareto frontier of the utility space, and defines a unique water level L .

The algorithm consists of several steps (see Table VI). In the first step we need to find what rule to apply: the Constraint Equal-Awards (CEA) rule or the Constraint Equal-Losses (CEL) rule. This can be resolved by determining whether there is a feasible allocation if the water level is above L_N . If so, the CEL rule is applied; otherwise the CEA rule. The decision is made by solving the following linear programming problem:

$$\begin{aligned} \mathbf{min} \quad & \sum_{n=1}^N \sum_{k=1}^K \alpha_{nk} \\ \mathbf{subject\ to:} \quad & \forall k \sum_{n=1}^N \alpha_{nk} \leq 1, \\ & \forall n, k \alpha_{nk} \geq 0, \\ & \forall n \sum_{k=1}^K \alpha_{nk} u_{nk} = \frac{u_n + d_n}{2} \end{aligned} \quad . \quad (8)$$

If there is a solution (the solution is in set \mathcal{S}_d), this implies that the water level L is above L_N and the allocation is according to the CEL rule; otherwise the water level is below L_n and the allocation is according to the CEA rule. We now explore these two cases.

Case A: CEA rule

All the players can gain at most half of $(u_n + d_2)$. Let p the smallest number such that $\{a_1, \dots, a_N\} \notin \mathcal{S}_d$, and

$$a_n = \begin{cases} \frac{u_n + d_n}{2} & n \leq p \\ \frac{u_p - d_p}{2} + d_n & p < n \leq N \end{cases} \quad . \quad (9)$$

This problem can be formulated as the following linear programming problem,

$$\begin{aligned} \mathbf{min} \quad & \sum_{n=1}^N \sum_{k=1}^K \alpha_{nk} \\ \mathbf{subject\ to:} \quad & \sum_{n=1}^N \alpha_{nk} \leq 1, \quad \forall k, \\ & \alpha_{nk} \geq 0, \quad \forall n, k \\ & \sum_{k=1}^K \alpha_{nk} u_{nk} = a_n, \quad \forall n \end{aligned} \quad . \quad (10)$$

Here, p can be found in a binary search. Now, the exact water level L has to be above $L_{p-1} + y$, but below the next level L_p . All players with an index greater than $p - 1$ will share the extra utility equally, and y

is the solution to the following linear programming problem

$$\begin{aligned}
 & \mathbf{max} \quad y \\
 & \mathbf{subject \ to:} \quad \sum_{n=1}^N \alpha_{nk} = 1, \quad \forall k, \\
 & \quad \alpha_{nk} \geq 0, \quad \forall n, k, \\
 & \quad \sum_{k=1}^K \alpha_{nk} u_{nk} = b_n, \quad \forall n
 \end{aligned} \tag{11}$$

and b_n is given by

$$b_n = \begin{cases} d_n + y & p = 1 \\ \frac{u_n + d_n}{2} & n < p < N \\ \frac{u_p - d_p}{2} + d_n + y & p \leq n \leq N \end{cases} . \tag{12}$$

The allocation to player n of commodity k is α_{nk} , where $\{\alpha_{nk}\}$ is the solution to equation (11).

Case B: CEL rule

All players lose at most half of $(u_n - d_n)$. Let p be the smallest p such that $\{a_1, \dots, a_N\} \in \mathcal{S}_d$, where a_n is given by

$$a_n = \begin{cases} \frac{u_n + d_n}{2} & n \leq p \\ u_n - \frac{u_p - d_p}{2} & p < n \leq N \end{cases} . \tag{13}$$

Similar to (10) with different values for the a_n 's, p can be found by a binary search.

Now, the exact water level has to be above $L_{2N-p} + y$, but below the next level L_{2N-p+1} . All players with an index equal or greater than p will share the extra utility equally, and y is the solution to the linear programming problem in equation (11), where b_n is given by

$$b_n = \begin{cases} \frac{u_n + d_n}{2} & n < p \\ u_n - \frac{u_p + p_n}{2} + y & p \leq n \leq N \end{cases} . \tag{14}$$

The allocation to player n of commodity k is α_{nk} , where α_{nk} is the solution to equation (11).

IV. DISCUSSION AND EXAMPLE

Any point on the Pareto frontier of set \mathcal{S} can be obtained by assigning a proper weight vector $\{w_1, \dots, w_n\}$, and solving the corresponding weighted max-min optimization problem. Zehavi et al. [16] proved that for a weighted max-min allocation problem of K commodities to N players, there is always a solution where at most $N - 1$ commodities are shared by more than one player. The Raiffa bargaining solution and the Aumann bankruptcy solution are Pareto optimal solutions that are located on the Pareto frontier. Therefore, in these solutions, the number of commodities that are shared by more than

TABLE VI
AUMANN BARGAINING SOLUTION FOR THE MULTI-COMMODITY CASE

<p>Initialization: Solve the linear programming problem in equation (8). If there is a solution then go to A, otherwise go to B.</p>
<p>A. All players gain at most half of $\frac{u_n + d_n}{2}$.</p> <ol style="list-style-type: none"> 1. Do a binary search to find the smallest p such that there is no solution to the linear programming in (10), and a_n are given in (9) 2. Solve the linear programming in (11), and b_n are given in (12) 3. b_n is the utility that is allocated to player n. <p>The commodities are allocated according to the αs.</p> <p>Exit</p>
<p>B. All players lose at most half of the debt.</p> <ol style="list-style-type: none"> 1. Do a binary search to find the smallest p such that there is a solution to the linear programming in (10), and a_n are given in (13) 2. Solve the linear programming in (11), and b_n are given in (14) 3. b_n is the utility that is allocated to player n. <p>The commodities are allocated according to the αs.</p>

one player is at most $N - 1$. Note that if the number of commodities is very large in comparison to the number of players, then it is easy to modified the allocation such that each player loses at most a single commodity it shares with others, and that loss is small when $N \gg K$.

Since the discovery of the Simplex Method in the 1940s, extensive work has been done on algorithms for solving Linear Programming, (LP). Large numbers of optimization algorithms have been developed including variants on Simplex Method, the Ellipsoid Method, and the Primal-Dual Interior-Point Method. Khachiyan [23] proved in 1979 that Linear Programming is polynomially solvable; namely, that an LP problem with rational coefficients, m inequality constraints and n variables can be solved in $O(n^3(n+m)L)$ arithmetic operations, where L^3 is the input length of the problem; i.e., the total binary length of the numerical data specifying the problem instance. In our case (the primal dual problem) we have $n = KN + 1$ variables and $m = K + N$ inequality constraints. Note that the matrix in our case is almost unimodular, and sparse, thus the worst case complexity is on the order of $O(K^6 N^5)$. Thus, the complexity of DRB is $O(JK^6 N^6)$, where J is the number of iterations, and the complexity of the Aumann bargaining solution

³ In our case $K(N-1)$ coefficients of the matrix A are zeros or ones; thus the value of L is bounded by $O(K^2 N)$, since $L = \sum_{i,j} \log_2(a_{i,j} + 1) + \log_2(nm) + (nm + m) = O(K^2 N)$

TABLE VII
SCENARIO I: THREE PLAYERS AND SEVEN COMMODITIES

Scenario I	Utility of commodities						
Player	1	2	3	4	5	6	7
1	3.0	4.7	2.3	8.4	1.9	2.2	1.7
2	8.7	6.2	18.4	8.6	3.7	18.1	19.6
3	3.9	9.0	14.3	20.8	9.2	21.1	24.9
Player	Raiffa- Commodities allocation						
1	0	1	0	0.844	0	0	0
2	1	0	1	0	0	0.58	0
3	0	0	0	0.156	1	0.42	1
Player	Aumann- Commodities allocation						
1	0	1	0	0.88	0	0	0
2	1	0	1	0	0	0.78	0
3	0	0	0	0.12	1	0.22	1
Player	Sum of utilities per player						
	Raiffa			Aumman			
1	11.7930			12.100			
2	37.5995			41.218			
3	46.1966			41.218			

TABLE VIII
SCENARIO II: THREE PLAYERS AND SEVEN COMMODITIES

Scenario II	Utility of commodities						
Player	1	2	3	4	5	6	7
1	8.4	8.7	3.0	0.1	0.2	0.5	0.3
2	0.3	0.2	18.5	12.1	19.6	0.5	0.2
3	0.2	0.7	10.5	0.1	1.0	31.1	30.4
Player	Raiffa- Commodities allocation						
1	1	1	0.126	0	0	0	0
2	0	0	0.792	1	1	0	0
3	0	0	0.082	0	0	1	1
Player	Aumann- Commodities allocation						
1	1	0.529	0	0	0	0	0
2	0	0	0.622	1	1	0	0
3	0	0.471	0.378	0	0	1	1
Player	Sum of utilities per player						
	Raiffa			Aumman			
1	17.4770			13.0017			
2	46.3581			43.2017			
3	62.3611			65.8017			

is $O(K^6 N^5)$. In practice, the algorithms converge faster than the worst case bound. A more extensive discussion on complexity can be found in [23].

Table VII presents a scenario with three players and seven commodities. Here, the utility of the commodity for each player is given in rows 3-5 of the table, and the ideal points of the players are (24.2, 83.3, 103.2), respectively. The allocations of the commodities for each player according to Raiffa are in rows 7-9, and the allocations according to Aumann are in rows 11-13. The final allocation for each player according to Raiffa and Aumann are given in rows 16-18. Similar results are shown for a different scenario in Table VIII, where the ideal points of the players are (21.2, 51.4, 74), respectively. However, the allocations for each player in scenario II are higher than in scenario I, due to the fact that most of the player's utility is concentrated in different commodities (scenario II). In the case of more than two players, allocating at least half of the ideal points to all players is sometimes not feasible (scenario I). In this case, the Aumann bargaining solution either allocates half of the claim to the player with the weakest ideal point half or allocates the utility equally among all the players.

V. SUMMARY AND CONCLUSIONS

The goal for this paper was to extend the discrete Raiffa bargaining solution and the Aumann bankruptcy solution to a multi-commodity game with N -players. We show that global bargaining solutions can be obtained by solving a sequence of linear programming problems. The complexity of the solution for DRB with M commodities is equivalent to solving N linear programming problems in each step, and the unallocated utilities decrease by a factor of e in each step.

APPENDIX A

SOLVING THE LINEAR PROGRAMMING PROBLEM FOR THE RAIFFA BARGAINING SOLUTION

Below we analyze the linear programming problem. To that end we formulate the KKT conditions [24]. Let $L(\alpha, \delta, \mu, \lambda)$ be the Lagrangian of the problem:

$$\begin{aligned}
L(\alpha, \delta, \mu, \lambda) = & - \sum_{k=1}^K \alpha_{nk}^{(n)} u_{nk} \\
& + \sum_{p=1, p \neq n}^N \lambda_p \left(\sum_{k=1}^K \alpha_{pk}^{(n)} u_{pk} - m_p^j \right) \\
& + \lambda_n \left(\sum_{k=1}^K \alpha_{nk}^{(n)} u_{nk} - m_n^j \right) \\
& + \sum_{k=1}^K \delta_k \left(\sum_{p=1}^N \alpha_{pk}^{(n)} - 1 \right) \\
& - \sum_{p=1}^N \sum_{k=1}^K \mu_{pk} \alpha_{pk}^{(n)}.
\end{aligned} \tag{15}$$

To better understand the problem, we first derive the KKT conditions [24]. Taking the derivative with respect to α_{nk}^j , and $\alpha_{pk}^j, p \neq n$ we obtain

$$\begin{aligned}
1. \quad & -u_{nk} + \lambda_n u_{nk} + \delta_k - \mu_{nk} = 0. \\
2. \quad & \lambda_p u_{pk} + \delta_k - \mu_{pk} = 0, \forall p \neq n,
\end{aligned} \tag{16}$$

with the complementarity conditions:

$$\begin{aligned}
1. \quad & \mu_{pk} \alpha_{pk}^{(n)} = 0, \mu, \delta \geq 0, \forall p. \\
2. \quad & \lambda_n \left(\sum_{k=1}^K \alpha_{nk}^{(n)} u_{nk} - m_n^j \right) = 0. \\
3. \quad & \delta_k \left(\sum_{p=1}^N \alpha_{pk}^{(n)} - 1 \right) = 0. \\
4. \quad & \sum_{k=1}^K \alpha_{pk}^{(n)} u_{pk} = m_p^j, \forall p \neq n.
\end{aligned} \tag{17}$$

There are several conclusions from these conditions:

- The allocation to player $p' \neq n$ is set such that $\sum_{k=1}^K \alpha_{pk}^{(n)} u_{pk} = m_{p'}^j$, and the ideal point of player n is $I_n = \sum_{k=1}^K \alpha_{nk}^{(n)} u_{nk}$.
- Based on (17.1) if $\alpha_{pk}^{(n)} > 0$, then $\mu_{pk} = 0$.
- If player n gets more than m_n^j , then $\lambda_n = 0$, based on (17.2), and if $\alpha_{nk}^{(n)} > 1$, then $\delta_k = u_{nk}$, due to ((16.1)), and ((17.1)).
- There are N thresholds, $\{\lambda_n\}_{n=1}^N$ that induce the allocation of the commodities and thresholds $\{\delta_k\}_{k=1}^K$.
- The threshold λ_p of player $p \neq n$ is set according to commodity k that was allocated to him, $-\lambda_p u_{pk} = \delta_k$ (see equation(16.2) and (17.1)) .
- Therefore, if a commodity k is shared by player p and player p' , then $\frac{u_{pk}}{u_{p'k}} = \frac{\lambda_{p'}}{\lambda_p}, \forall p, p' \neq n$.
- If a commodity k is shared by players p and player n , then $-u_{pk} \lambda_p = u_{nk}$.
- If a commodity k is shared by player p and not by player p' , then $\frac{u_{pk}}{u_{p'k}} > \frac{\lambda_{p'}}{\lambda_p}$.

APPENDIX B

THE LINEAR PROGRAMMING SOLUTION FOR 2 PLAYERS

For the two player case the linear programming problem can be dramatically simplified, and we provide an $O(K \log_2 K)$ complexity algorithm (K is the number of commodities). We show that the two players share at most a single commodity, no matter what the ratios between the users are. To that end let, $\alpha_{1k} = \alpha_k$, and $\alpha_{2k} = 1 - \alpha_k$.

We want to solve the following optimization problem:

$$L(\boldsymbol{\alpha}, \boldsymbol{\delta}, \boldsymbol{\mu}, \boldsymbol{\lambda}) = - \sum_{k=1}^K (1 - \alpha_k) u_{2k} - \sum_{k=1}^K \mu_k \alpha_k + \lambda \left(\sum_{k=1}^K \alpha_k u_{1k} - u_1 \right). \quad (18)$$

To better understand the problem, we first derive the KKT conditions [24]. Taking the derivative with respect to α_k , we obtain

$$u_{2k} + \lambda u_{1k} - \mu_k = 0. \quad (19)$$

with the complementarity conditions:

$$\begin{aligned} 1. \quad & \sum_{k=1}^K \alpha_{1k} u_{1k} = u_1, \\ 2. \quad & \mu_k \alpha_k = 0, \quad \mu \geq 0. \end{aligned} \quad (20)$$

Based on (19)-(20), we can easily see that the Lagrange multipliers in (20) satisfy the following conclusions :

1. $\mu_{nk} = 0$, if $\alpha_k > 0$, $\forall k$ (see (20.2)).
2. If $0 < \alpha_{1k}^j < 1$, then the players share a commodity p if $\frac{u_{2p}}{u_{1p}} = -\lambda$ (see (19.2)).
3. Commodity p is assigned to player 2 if $\frac{u_{2p}}{u_{1p}} > -\lambda$.
4. Commodity p is assigned to player 1 if $\frac{u_{2p}}{u_{1p}} < -\lambda$.
5. $\sum \alpha_{1k} u_{1k} = u_1$.

Assuming that a feasible solution exists and that the commodities are sorted in decreasing order according to the ratio $L(k) = \frac{u_{1k}}{u_{2k}}$, it follows from the KKT conditions that the allocation is made according to the following rules

- 1) The ideal point of player 1 is I_1 given by

$$I_1(u_2) = \sum_{k=1}^{p-1} u_{1k} + \alpha_p u_{1p}, \quad (21)$$

where p and α_p are set such that

$$u_2 = \sum_{k=p}^K u_{2k} - \alpha_p u_{2p}. \quad (22)$$

2) Similarly, the ideal point of player 2 is $I_2(u_1)$ is given by

$$I_2(u_1) = \sum_{k=p}^K u_{2k} - \alpha_p u_{2p}, \quad (23)$$

where p and α_p are set such that

$$u_1 = \sum_{k=1}^{p-1} u_{1k} + \alpha_p u_{1p}. \quad (24)$$

Therefore, no more than one commodity can be shared by the two players. The algorithm for computing the ideal point of player 1 is as follows. Let $L_k = \frac{u_{1k}}{u_{2k}}$ be the ratio between the utilities of commodity k . We can sort the commodities in decreasing order according to L_k . If all the values of L_k are distinct then there is at most a single commodity that has to be shared between the two players. Since only one commodity satisfies equation (24), we denote this commodity as k_s , then all the commodities $1 \leq k < k_s$ will be allocated to player 1, while all the commodities $k_s < k \leq K$ will be allocated to player 2. The commodity k_s must be shared accordingly between the players. The complexity of this algorithm is at most $O(K \log K)$, due to the sorting operation. For the Raiffa bargaining solution only the sorting operation has to be done once at the beginning. The complexity of computing the next disagreement point is on the order of $O(K)$.

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